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DETERMINATION OF THE ULTIMATE STATES OF ELASTOPLASTIC BODIES

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The equations of quasistatic deformation of elastoplastic bodies are considered in a geometrical linear formulation. After discretization of the equations with respect to spatial variables by the finite-element method, the problem of determining equilibrium configurations reduces to integration of a system of nonlinear ordinary differential equations. In the ultimate state of a body of an ideal elastoplastic material, the matrix of the system degenerates and the problem becomes singular. A regularization algorithm for determining solutions of the problems for the ultimate states of bodies is proposed. Numerical solutions of test problems of determining the ultimate loads and equilibrium configurations for ideal elastoplastic bodies confirm the reliability of the regularization algorithm proposed.

INTRODUCTION

In the solution of problems of quasistatic deformation of ideal elastoplastic bodies [1, 2], *ultimate states*, i.e., equilibrium configurations in which the strains tend to infinity, occur for certain values of external forces. The corresponding load is called the *ultimate load*. Analytical solutions of some problems and determination of the corresponding ultimate loads are given in [2].

The finite-element method (FEM) is a universal method for solving problems of elastoplastic deformation for bodies of arbitrary geometry [3–5]. After discretization of the basic system of differential equations by the FEM, the equilibrium configurations of the body can be determined by step-by-step integration of a nonlinear system of ordinary differential equations (ODE). In the ultimate state of the body, the matrix of the system degenerates. In the standard procedures of step-by-step integration of an ODE system, the external load is used as a deformation parameter. To determine the equilibrium states in the neighborhood of the ultimate loads, it is necessary to use small increments in the load [6]. In this case, the iterative processes of refining the solution converge poorly or diverge. A more reliable algorithm for determining the equilibrium configurations of the body in the neighborhood of the ultimate loads is based on the introduction of external load into the number of unknowns and the use of arc length of the integral curve as the deformation parameter [7, 8].

In the present paper, the procedure proposed by Bathe and Dvorkin [8] is supplemented by regularization of the system of ODE. This procedure of determining the equilibrium configurations is implemented in the computer program PIONER [9]. Its efficiency is demonstrated by solving some test problems.

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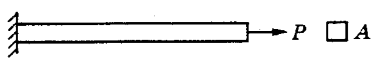

Fig. 1

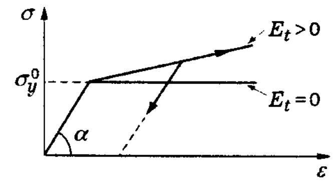

Fig. 2

1. REGULARIZATION OF THE ONE-DIMENSIONAL PROBLEM

We solve a model problem to show difficulties that arise in problems of deformation of ideal elastoplastic bodies in the ultimate states and consider the regularization algorithm that is used below to solve two- and three-dimensional problems.

We consider the uniaxial uniform deformation of a bar with cross-sectional area A stretched or compressed by axial force P (Fig. 1). The material of the bar is assumed to be elastoplastic with the bilinear stress–strain diagram shown in Fig. 2 (σ and ε are the axial stress and strain, respectively, $\sigma_y^0 > 0$ is the initial value of the yield stress, $\tan\alpha = E > 0$ is Young's modulus, and $E_t \geqslant 0$ is the tangent modulus). Using the theory of plastic flow with isotropic hardening, we write the constitutive relation for the bar material in the form

$$\dot{\sigma} = b\dot{\varepsilon}, \tag{1.1}$$

where

$$b \equiv \begin{cases} E, & |\sigma| < \sigma_y \quad \text{or} \quad |\sigma| = \sigma_y, \quad \sigma\dot{\varepsilon} \leqslant 0, \\ E_t, & |\sigma| = \sigma_y, \quad \sigma\dot{\varepsilon} > 0. \end{cases} \tag{1.2}$$

Here $\sigma_y \equiv \max\limits_{0\leqslant\tau\leqslant t}\{|\sigma(\tau)|, \sigma_y^0\}$ is the current value of the yield stress, t is a monotonically increasing deformation parameter, whose initial value is $t = 0$, and the dot denotes differentiation with respect to t.

We write the rate equilibrium equation:

$$\dot{\sigma} = \dot{P}/A. \tag{1.3}$$

From (1.1) and (1.3), we obtain the nonlinear ODE

$$k\dot{\varepsilon} = \dot{P}, \tag{1.4}$$

subject to the initial condition

$$\varepsilon = 0 \quad \text{for} \quad t = 0. \tag{1.5}$$

Here, we introduce the *tangent coefficient of stiffness* (analog of a tangent stiffness matrix [3])

$$k(\sigma, \dot{\varepsilon}) \equiv bA.$$

Problem (1.4), (1.5) governs the deformation of an elastoplastic bar. Equation (1.4) is an analog of the system of equations obtained by FEM discretization of the differential equations governing the deformation of an elastoplastic body.

The equilibrium configurations of the bar are determined by integrating Eq. (1.4). For the monotonic plastic deformation of the rod, the equilibrium configurations in the (ε, σ) plane correspond to the uniaxial strain diagram shown in Fig. 2.

The solvability of problem (1.4), (1.5) depends on the value of the tangent modulus E_t:

— $E_t > 0$ (*strain hardening elastoplastic material*). Problem (1.4), (1.5) is regular for both elastic ($k = EA$) and elastoplastic ($k = E_tA$) deformation. The solution of the problem is unique. The ultimate states do not occur;

— $E_t = 0$ (*ideal elastoplastic material*). Problem (1.4), (1.5) is regular for elastic deformation ($k = EA$) and is singular for elastoplastic deformation ($k = 0$). In the latter case, for $\dot{P} \neq 0$, solutions do not exist, and for $\dot{P} = 0$, a solution exists and it is not unique. The ultimate state occurs for $P_{\lim} = \sigma_y^0 A$.

For an elastoplastic bar of strain hardening material, problem (1.4), (1.5) can be solved assuming that the force P is the deformation parameter ($\dot{P} = 1$). For a bar of an ideal elastoplastic material, the force P cannot be used as the deformation parameter when deformation occurs in the inelastic region since problem (1.4), (1.5) has no solution for $\dot{P} = 1$. Thus, within the framework of the standard formulation of the problem of deformation of an elastoplastic body of the form (1.4), (1.5) under a specified external force, the equilibrium configurations cannot be obtained in the ultimate state.

In the new formulation of problem (1.4), (1.5), it is assumed that the external force is unknown. We introduce the unknown parameter λ such that

$$P = \lambda P_0,$$

where $P_0 = \text{const} \neq 0$. We now have the two desired functions ε and λ. Equation (1.4) is rewritten in the form

$$k\dot{\varepsilon} - \dot{\lambda} P_0 = 0. \tag{1.6}$$

Equation (1.6) is solved in conjunction with the *control equation*

$$\dot{\varepsilon}^2 + \dot{\lambda}^2 = 1, \tag{1.7}$$

for which the deformation parameter t is the arc length of the integral curve in the space (ε, λ) [7, 8]. Equations (1.6) and (1.7) are supplemented by the initial conditions

$$\varepsilon = \lambda = 0 \quad \text{for} \quad t = 0. \tag{1.8}$$

In the ultimate state of the bar, system (1.6), (1.7) becomes

$$\dot{\lambda} = 0, \qquad \dot{\varepsilon} = \pm 1. \tag{1.9}$$

The plus and minus signs on the right side of the second equation correspond to tension and compression of the bar, respectively. The choice of the arc length as the deformation parameter implies that in the ultimate state, the absolute value of the strain becomes the deformation parameter [see the second equality in (1.9)]. In contrast to problem (1.4), (1.5), which becomes singular in the ultimate state, problem (1.8), (1.9) is regular and solvable. The nonlinear equation (1.7) reduces to the second equation (1.9), which is linear since $\dot{\lambda} = 0$. The latter equality is the solvability condition for Eq. (1.6) with $k = 0$. Generally (in the solution of problems of large dimension), equations of the form (1.7) must be linearized and the replacement of a system of the form (1.4) by a system of the form (1.6) changes the structure of the matrix of the basic system of equations.

We consider another regularization algorithm, which is more suitable for numerical methods. We solve system (1.6), (1.7) in two stages. In the first stage, we solve Eq. (1.6), which after introduction of the new variable

$$x \equiv \dot{\varepsilon}/\dot{\lambda} \tag{1.10}$$

takes the form

$$kx = P_0. \tag{1.11}$$

Equation (1.11) is similar to Eq. (1.4) and is singular too. We regularize this equation, replacing $E_t = 0$ by $E'_t > 0$ ($E'_t \ll E$), i.e., the ideal elastoplastic material of the bar is replaced by a strain-hardening material with a small (compared to Young's modulus) tangent modulus. For the new material, we introduce the notation $k' \equiv b'A$, where b' is determined by replacing E_t by E'_t. Thus, instead of the singular equation (1.11), we consider the regularized equation

$$k'x = P_0. \tag{1.12}$$

In the second stage, we determine $\dot{\lambda}$ from the control equation (1.7):

$$\dot{\lambda} = \pm \frac{1}{\sqrt{1 + x^2}}, \tag{1.13}$$

and $\dot{\varepsilon}$ from (1.10): $\dot{\varepsilon} = x\dot{\lambda}$. The sign on the right side of (1.13) is chosen from the condition of greater smoothness of the integral curve in the (ε, λ) space.

Since, the basic equation (1.11) is replaced by (1.12) as a result of regularization, the original problem should be solved using an iterative procedure to refine the solution.

2. REGULARIZATION OF A SYSTEM OF FINITE-ELEMENT EQUATIONS

The above regularization procedure for the one-dimensional problem forms the basis of a numerical algorithm for the deformation analysis of ideal elastoplastic bodies in the ultimate states.

After FEM discretization in the spatial variables, the system of equations governing the quasistatic deformation of an elastoplastic body is written as [3–5]

$$K\dot{\boldsymbol{U}} = \dot{\boldsymbol{R}}. \tag{2.1}$$

Here K is the symmetric tangent stiffness matrix, $\boldsymbol{U}$ is the vector of nodal displacements, $\boldsymbol{R}$ is the vector of external forces, and the remaining notation coincides with the one adopted in Sec. 1. System (2.1) is supplemented by the initial conditions

$$\boldsymbol{U} = \boldsymbol{U}_0 \quad \text{for} \quad t = 0. \tag{2.2}$$

The constitutive relations used to calculate the matrix K have the form

$$\dot{\sigma}_{ij} = C_{ijkl}\dot{\varepsilon}_{kl} \qquad (C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}), \tag{2.3}$$

where σ_{ij} and ε_{kl} are components of the stress and strain tensors, respectively, and C_{ijkl} are components of the tensor of rank four of the constitutive relations; the indices run from 1 to 3, and summation is performed over repeated indices. For the theory of plastic flow with isotropic strain hardening, the components of the tensor of the constitutive relations are as follows [10]:

$$C_{ijkl} = \frac{E}{1+\nu}\Big[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\,\delta_{ij}\delta_{kl} - c\,\frac{d\sigma'_{ij}\sigma'_{kl}}{1+\nu+2dJ_2}\Big], \tag{2.4}$$

where

$$c = \begin{cases} 0, & f_y < 0 \quad \text{or} \quad f_y = 0, \quad \sigma'_{ij}\dot{\varepsilon}_{ij} \leqslant 0, \\ 1, & f_y = 0, \quad \sigma'_{ij}\dot{\varepsilon}_{ij} > 0; \end{cases} \tag{2.5}$$

$\sigma'_{ij} \equiv \sigma_{ij} - \delta_{ij}\sigma_{kk}/3$ are the components of stress tensor deviator, ν is Poisson's ratio, $f_y \equiv \sqrt{3J_2} - \sigma_y$ is the von Mises yield function, and δ_{ij} are the Kronecker symbols; the second invariant of the stress tensor deviator J_2 and the current yield point σ_y are determined by the equations

$$J_2 \equiv \sigma'_{ij}\sigma'_{ij}/2, \qquad \sigma_y \equiv \max_{0\leqslant\tau\leqslant t}\{\sqrt{3J_2(\tau)}, \sigma_y^0\}.$$

The function introduced in (2.4) has the form

$$d(J_2) \equiv \frac{3}{4J_2}\Big(\frac{E}{E_t} - 1\Big).$$

For an ideal elastoplastic material ($E_t = 0$), the replacement

$$c\,\frac{d\sigma'_{ij}\sigma'_{kl}}{1+\nu+2dJ_2} \;\to\; c\,\frac{\sigma'_{ij}\sigma'_{kl}}{2J_2} \tag{2.6}$$

must be made on the right side of (2.4). The constitutive relations (2.3), (2.4), and (2.6) generalize relations (1.1) and (1.2) for the uniaxial deformation of a bar.

By virtue of (2.5), we have $K = K(\dot{\boldsymbol{U}})$, and, hence, (2.1) and (2.2) constitute the Cauchy problem for the system of nonlinear ODE.

We assume that the external-force vector depends only on the parameter λ, which characterizes the intensity of the external forces:

$$\boldsymbol{R} = \lambda\boldsymbol{R}_0. \tag{2.7}$$

Here the vector $\boldsymbol{R}_0$ characterizes the external-force distribution. With allowance for (2.7), system (2.1) becomes

$$K\dot{\boldsymbol{U}} = \dot{\lambda}\boldsymbol{R}_0. \tag{2.8}$$

The ultimate load (the parameter λ takes the value λ_{lim}) corresponds to the degenerate matrix K. The latter is possible only for an ideal elastoplastic body. The standard schemes of integration of the nonlinear problem (2.8), (2.2) fail in the neighborhood of the ultimate loads.

We regularize the problem with the degenerate (or almost degenerate) matrix K according to the scheme considered in Sec. 1. The parameter λ is assumed to be unknown. System (2.8) is supplemented by a new (control) equation that corresponds to the choice of the arc length of the integral curve in the $(\boldsymbol{U}, \lambda)$ space as the parameter t:

$$\dot{\lambda}^2 + \dot{\boldsymbol{U}}^{\mathrm{t}}\dot{\boldsymbol{U}} = 1 \qquad (2.9)$$

(the superscript "t" denotes transposition of the column vector).

System (2.8), (2.9) is solved for $(\dot{\boldsymbol{U}}, \dot{\lambda})$ in two stages. First, an auxiliary system of algebraic equations is solved for the vector $\hat{\boldsymbol{U}}$

$$K\hat{\boldsymbol{U}} = \boldsymbol{R}_0 \quad \text{or} \quad K'\hat{\boldsymbol{U}} = \boldsymbol{R}_0, \qquad (2.10)$$

then $\dot{\lambda}$ and $\dot{\boldsymbol{U}}$ are determined from the vector $\hat{\boldsymbol{U}}$ by the formulas

$$\dot{\lambda}_{1,2} = \pm 1 \Big/ \sqrt{1 + \hat{\boldsymbol{U}}^{\mathrm{t}}\hat{\boldsymbol{U}}}, \qquad \dot{\boldsymbol{U}} = \hat{\boldsymbol{U}}\dot{\lambda}. \qquad (2.11)$$

In the determination of $\dot{\lambda}$ in (2.11), the sign is chosen from the condition of greater smoothness of the integral curve in the $(\boldsymbol{U}, \lambda)$ space [11]. The matrix K' in (2.10) is obtained from the matrix K by replacement of the tangent modulus $E_t = 0$ by $E_t' > 0$ ($E_t' \ll E$), i.e., the ideal elastoplastic material is replaced by a material with small strain hardening.

Problem (2.10), (2.11), (2.2) (with allowance for $\lambda = 0$ for $t = 0$) is solved by stepwise integration with iterative refinement of the solution by the Crisfield method, which is considered in detail in [8, 12]. It should be noted that in the iterative solution of problems of the deformation of ideal elastoplastic bodies, the internal-force vector $\boldsymbol{F}$ [3] is calculated using the tangent modulus $E_t = 0$. The modified value $E_t' > 0$ is used only to regularize the problem when the trial vector $\hat{\boldsymbol{U}}$ is determined.

3. NUMERICAL IMPLEMENTATION OF THE REGULARIZATION ALGORITHM

The above regularization algorithm for an ODE system with a degenerate matrix is implemented in the computer program PIONER [9] to solve two- and three-dimensional problems of deformation of ideal elastoplastic bodies. We consider solutions of some test problems obtained using this program.

3.1. Tension of a Bar. We consider a numerical solution of the problem formulated in Sec. 1. A bar with the geometrical parameters given in Fig. 3a is loaded by a tensile force P. The bar material is ideal elastoplastic with material constants $E = 1$ GPa, $\nu = 0.3$, and $\sigma_y^0 = 10$ MPa.

The first solution was obtained with the use of the bar (one-dimensional) model. The bar was divided into 10 two-node elements. The constitutive relations of this model have the form (1.1). We were unable to solve the problem without modification (without replacing $E_t = 0$ by $E_t' > 0$) since the auxiliary problem (2.10) with the singular matrix K and nonzero right side has no solution. However, the regularized problem (where $E_t = 0$ is replaced by $E_t' = 0.01E$) was solved. The uniaxial stress–strain diagram was reproduced in the numerical solution with high accuracy (points in Fig. 3b).

The second solution was obtained with the use of a two-dimensional model (plane stress) of the bar. The bar was divided into 10 four-node elements (Fig. 3a). To calculate the tangent stiffness matrix K, we used constitutive relations of the form (2.3) and (2.4). If the calculations were exact, the matrix K would be singular. However, because of approximation and calculation errors, the matrix K in system (2.10) proves nonsingular (uncontrolled regularization of the problem). The matrix K' is nonsingular with replacement of $E_t = 0$ by $E_t' = 0.01E$ (controlled regularization). The uniaxial deformation curve in Fig. 3c was reproduced adequately for both the controlled and uncontrolled regularization of the problem. The following two problems were solved for the controlled regularization ($E_t = 0$ was replaced by $E_t' = 0.01E$).

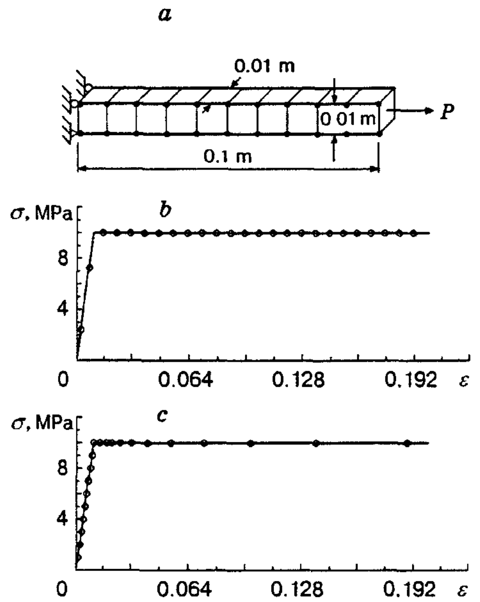

Fig. 3

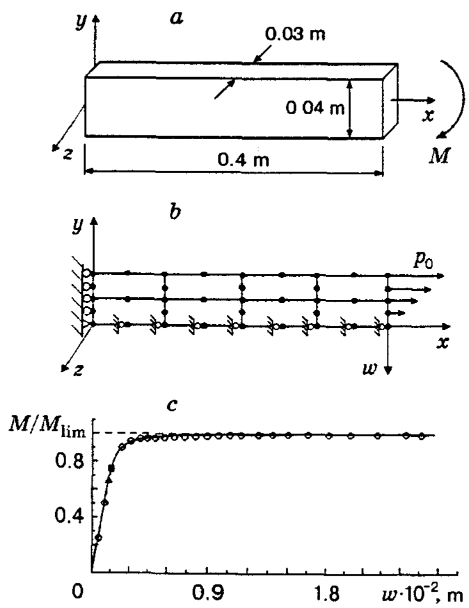

Fig. 4

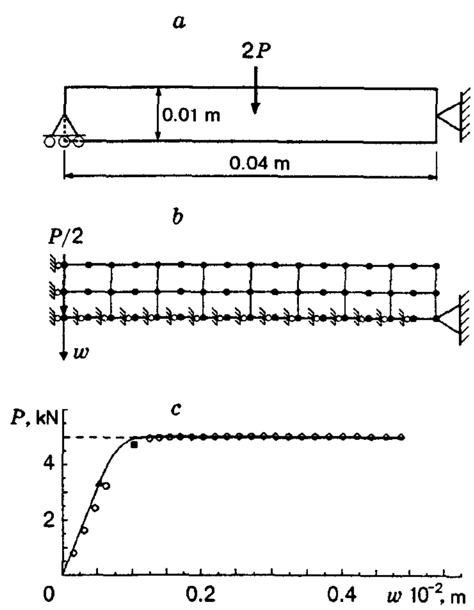

Fig. 5

3.2. "Pure" Bending of a Beam. A beam is bent by moment M. Owing to the symmetry, half of the beam is considered (Fig. 4a). The beam material is assumed to be ideal elastoplastic ($E = 3000$ GPa, $\nu = 0.3$, and $\sigma_y^0 = 1$ GPa). Since in a geometrical linear formulation, the beam axis is not deformed under the bending moment, numerical analysis was performed for a quarter of the beam. A two-dimensional model (plane stress) is used and the beam is divided into eight-node elements (Fig. 4b). The moment at the beam's end is modeled by a linear pressure distribution with a maximum value $p_0 = 75$ MPa (Fig. 4b), which is used to determine the vector $\boldsymbol{R}_0$ in (2.7). This pressure distribution models the pure bending of the beam in an elastic formulation; however, in an elastoplastic formulation, pure bending is modeled with an error due to the nonlinear distribution of the longitudinal stress over the cross section of the beam.

Kachanov [2] considered the problem in a three-dimensional formulation and obtained an analytical solution by the Saint Venant semiinverse method. Figure 4c shows the moment applied on the end of the beam M versus the normal deflection of the beam end w, where $M_{\rm lim}$ is the ultimate moment calculated in [2] to be $12 \cdot 10^3$ N $\cdot$ m. The solid curve refers to the analytical solution and the points refer to the equilibrium configurations obtained by numerical integration of the equations. The equilibrium configurations in which the plastic-deformation regions occur for the first time are indicated in Fig. 4c by the triangle and square for the analytical and numerical solutions, respectively. Good agreement between the solutions is observed.

3.3. Transverse Bending of a Beam. We consider the problem of a beam loaded by a point force applied at its center (Fig. 5a). The beam material is elastoplastic ($E = 20$ GPa, $\nu = 0$, and $\sigma_y^0 = 400$ MPa). An analytical solution of the problem is given in [2]. Finite-element modeling of a quarter of the beam (Fig. 5b) is similar to that considered in Sec. 3.2. Figure 5c shows the normal mid-span deflection as a function of the applied force. There is good agreement between the numerical and analytical solutions (the notation is the same as in Fig. 4c). According to the analytical solution [2], the ultimate value of the halved force is $P_{\rm lim} = 5$ kN.

CONCLUSIONS

An algorithm is proposed for the regularization of problems of deformation of ideal elastoplastic bodies in the ultimate states. This algorithm is a modification of the standard numerical algorithm (the external load is assumed to be specified) and consists of two stages. In the first stage, the intensity parameter of the external load is considered unknown and the arc length of the integral curve in the $(\boldsymbol{U}, \lambda)$ space is taken to be the deformation parameter. In the second stage, the degenerate matrix of the system of equations for an ideal elastoplastic material is replaced by a matrix that corresponds to a material with isotropic strain hardening. The error due to this replacement is eliminated by iterative refinement of the solution. Numerical experiments show that some problems can be solved without the second stage of regularization since calculations yield an almost singular rather than singular tangent stiffness matrix. However, the uncontrolled regularization can lead to an unreliable solution of the problem. Therefore, it is recommended to employ the regularization algorithm proposed in the present paper.

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REFERENCES

1. B. D. Annin and G. P. Cherepanov, *Elastic-Plastic Problems*, ASME Press, New York (1988).
2. L. M. Kachanov, *Foundations of the Theory of Plasticity*, North-Holland, Amsterdam–London (1971).
3. K.-J. Bathe, *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey (1982).
4. O. C. Zienkiewicz and R. L. Taylor, *The Finite Element Method*, McGraw-Hill, London (1991).
5. S. N. Korobeinikov, "Solution of two-dimensional geometrical and physical nonlinear problems by the finite element method," in: *Numerical Methods of Solving Problems of the Theory of Elasticity and Plasticity*, Proc. X All-Union Conf. (Krasnoyarsk, February 23–27, 1987), Novosibirsk (1988), pp. 134–140.
6. A. N. Blinov, "Boundedness of the average stress for small elastoplastic strains," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 92, Novosibirsk (1989), pp. 39–44.
7. V. I. Shalashilin and E. B. Kuznetsov, *The Parameter-Control Method of Continuation of Solution and the Best Parametrization* [in Russian], Éditorial URSS, Moscow (1999).

8. K.-J. Bathe, E. N. Dvorkin, "On the automatic solution of nonlinear finite element equations," *Comput. Struct.*, **17**, Nos. 5/6, 871–879 (1983).
9. S. N. Korobeinikov, V. P. Agapov, M. I. Bondarenko, and A. N. Soldatkin, "The general purpose nonlinear finite element structural analysis program PIONER," in: *Proc. of the Int. Conf. on Num. Methods and Appl.* (Sofia, August 22–27 1988), Publ. House of the Bulgarian Acad. of Sci., Sofia (1989), pp. 228–233.
10. M. Kleiber, *Incremental Finite Element Modelling in Non-Linear Solid Mechanics*, Ellis Horwood, Chichester (1989).
11. T. Sokol and M. Witkowski, "The equilibrium path determination in non-linear analysis of structures," in: *Advances in Non-Linear Finite Element Methods*, Proc. of the 2nd Int. Conf. on Comput. Structures Technol. (Athens, August 30–September 1, 1994), Civil-Comp Press, Edinburgh (1994), pp. 35–45.
12. S. N. Korobeinikov, "Application of the finite element method to solution of nonlinear problems of deformation and buckling of atomic lattices," Preprint No. 1-97, Lavrent'ev Inst. of Hydrodynamics, Sib. Div., Russian Acad. of Sci., Novosibirsk (1997).